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THE LYAPUNOV–POINCARÉ METHOD IN THE THEORY OF PERIODIC MOTIONS†‡

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The problem of the periodic motions of a system with a small parameter is solved. The non-rough cases, when the problem cannot be solved by a generating system obtained for a zero value of the small parameter, are investigated. Lyapunov's idea of using a new generating system which already contains the small parameter is systematically developed. Systems of general form, inverse systems and systems close to inverse are investigated.

1. FORMULATION OF THE PROBLEM

Consider a fairly smooth autonomous or 2π-periodic system

$$\mathbf{x}^{\bullet} = \mathbf{X}(\mathbf{x}, t) + \mu\mathbf{X}_1(\mu, \mathbf{x}, t), \quad \mathbf{x} \in \mathbf{R}^m \tag{1.1}$$

with a small parameter μ. We will assume that when $\mu = 0$ the generating system

$$\mathbf{x}^{\bullet} = \mathbf{X}(\mathbf{x}, t) \tag{1.2}$$

admits of a periodic solution $\mathbf{x} = \varphi(t)$, and we will investigate the issue on the existence in system (1.1) when $\mu \neq 0$ of a periodic solution which changes into the solution $\varphi(t)$ as $\mu \to 0$.

The formulation of this problem is due to Poincaré [1], and the main method of solving it is Poincaré's method [1, 2], which was initially proposed for problems of celestial mechanics [1] and developed in more detail for analytic systems of general form (1.1) [2]. Two cases arise when solving the problem: (a) a rough case, when the property of system (1.1) of having a periodic solution is determined solely by the generating system, and (b) a non-rough case, when, to solve the problem, it is necessary to consider the perturbation $\mu\mathbf{X}_1$. In systems of the general form (1.1) the Poincaré-isolated case [2] is rough.

Generating system (1.1) belongs to a certain class K, for example, it is conservative, a Lyapunov system, etc. The perturbations are also considered from a certain class P. System (1.2) from class K may be rough in the sense of the property of possessing a periodic solution for perturbations from class P_1 and of being non-rough for perturbations from class P_2. The classes K and P are defined by the content of the specific problem being investigated. In the theoretical scheme and when solving applied problems it is correct to assume a class P identical with the class K, or to assume that the perturbations $\mu\mathbf{X}_1$ are of more general form.

In non-rough cases it is usual to analyse a perturbation of finite order (as a rule, the first) with respect to the small parameter [2]. Thereby, one in fact (implicitly) constructs a new generating system, already containing the small parameter. The idea of choosing such a generating system (in Hill's problem) is due to Lyapunov [3], but was not subsequently developed. Lyapunov's approach, in well-known cases, leads to the same results as Poincaré's method. This approach turns out to be preferable when analysing a number of non-rough cases, in particular, for oscillatory systems of standard form. This enables us to formulate the conditions for a periodic solution to exist in a form assumed in averaging methods.

We will formulate the problem of the systematic development of Lyapunov's idea in the problem of periodic solutions of a system with a small parameter in non-rough cases. We will consider systems of general form, inverse systems and systems close to inverse.

2. THE CHOICE OF GENERATING SYSTEM

Consider the autonomous or 2π-periodic system

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$$\mathbf{x}^{\bullet} = \mathbf{X}(\varepsilon, \mathbf{x}, t) + \mu \mathbf{X}_1(\varepsilon, \mu, \mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^m \tag{2.1}$$

with two small parameters ε and μ. We will assume that when $\mu = 0$ system (2.1) allows of a $2T^*(\varepsilon)$-periodic solution $\mathbf{x} = \varphi(\varepsilon, t)$, $\varphi(\varepsilon, 0) = \mathbf{x}^*$ ($T^*(\varepsilon) = \pi$ for 2π-periodic system (2.1)). We will investigate in which class of functions $\mu = \mu(\varepsilon)$ the problem of the existence in (2.1) of a periodic motion can be solved by the generating system obtained from (2.1) with $\mu = 0$. The problem of choosing the generating system which contains the small parameter is thereby obviously solved.

We will denote by $\mathbf{x}(\varepsilon, \mu, x_1^\circ, \ldots, x_m^\circ, t)$ the solution of system (2.1) with initial conditions $x_1^\circ, \ldots, x_m^\circ$. Then, the necessary and sufficient conditions for $2T$-periodicity of the solution ($T = \pi$ for a 2π-periodic system) have the form

$$x_s(\varepsilon, \mu, x_1^\circ, \ldots, x_m^\circ, 2T) - x_s^\circ = 0 \quad (s = 1, \ldots, m) \tag{2.2}$$

and consist of a system of m functional equations in $x_1^\circ, \ldots, x_m^\circ$ (and T in the case of an autonomous system). This system is compatible for $\mu = 0$ and has a solution $\mathbf{x}^\circ = \mathbf{x}^*$, $T = T^*$. Hence, introducing the increments $\Delta\mathbf{x}^\circ = \mathbf{x}^\circ - x^*$, $\Delta T = T - T^*$, system (2.2) can be represented in the form

$$\sum_{j=1}^{m} [F_{sj}^*(\varepsilon) + F_{sj}]\Delta x_j^\circ + [G_s^*(\varepsilon) + G_s]\Delta T + \mu \Psi_s(\varepsilon, \mu, \Delta\mathbf{x}^\circ, \Delta T) = 0 \quad (s = 1, \ldots, m) \tag{2.3}$$

The functions F_{sj} and G_s depend on ε, $\Delta\mathbf{x}^\circ$, ΔT and vanish when $\Delta\mathbf{x}^\circ = 0$, $\Delta T = 0$.

1. *A periodic system*. Here $\Delta T = 0$. When rank $\| F_{sj}^*(0) \| = m$ we have the rough case with respect to ε and μ, while the solution $\mathbf{x} = \varphi(0, t)$ of system (2.1) when $\varepsilon = \mu = 0$ is continued with respect to the parameters ε and μ [2].

Consider the non-rough cases when rank $\| F_{sj}^*(0) \| = m - k$ ($k > 0$). Here, without loss of generality, we will assume that

$$F_{\alpha j}^*(\varepsilon) = \varepsilon[a_{\alpha j} + F_{\alpha j}^\circ(\varepsilon)] \quad (\alpha = 1, \ldots, k), \quad \Delta x_s^\circ = \varepsilon \xi_s \quad (s = 1, \ldots, m)$$

where $a_{\alpha j}$ is independent of ε and $F_{\alpha j}^\circ(0) = 0$. Then, system (2.3) takes the form

$$\begin{gathered}\sum_{j=1}^{m} \left[a_{\alpha j} + F_{\alpha j}^\circ(\varepsilon) + \frac{1}{\varepsilon} F_{\alpha j}(\varepsilon, \varepsilon\xi)\right]\xi_j + \frac{\mu}{\varepsilon^2}\Psi_\alpha(\varepsilon, \mu, \varepsilon\xi) = 0 \quad (\alpha = 1, \ldots, k) \\ \sum_{j=1}^{m} [F_{\beta j}^*(\varepsilon) + F_{\beta j}(\varepsilon, \varepsilon\xi)]\xi_j + \frac{\mu}{\varepsilon}\Psi_\beta(\varepsilon, \mu, \varepsilon\xi) = 0 \quad (\beta = k+1, \ldots, m)\end{gathered} \tag{2.4}$$

If $\mu = O(\varepsilon^{2+\sigma})$, $\sigma > 0$, system (2.4) with $\varepsilon = 0$ has a unique solution $\varepsilon_s = 0$ ($s = 1, \ldots, m$) provided that

$$\operatorname{rank}\begin{Vmatrix} a_{\alpha j} \\ F_{\beta j}^*(0) \end{Vmatrix} = m \tag{2.5}$$

When condition (2.5) is satisfied, system (2.4) is also compatible for sufficiently small $\varepsilon \neq 0$, where $\xi_s = O(\varepsilon^\sigma)$ ($s = 1, \ldots, m$), if $\Psi_s(0, 0, 0) \neq 0$ ($s = 1, \ldots, m$). Hence, in general, we have $\Delta\mathbf{x}^\circ = O(\varepsilon^{1+\sigma})$.

Similar discussions also hold in the case when the non-degeneracy of the matrix $\| F_{sj}^*(\varepsilon) \|$ is verified by terms of order ε^ν inclusive.

Theorem 1. Suppose a 2π-periodic solution exists in the 2π-periodic system (2.1) when $\mu = 0$. Then, in any class of functions $\mu = O(\varepsilon^{2\nu+\sigma})$, $\sigma > 0$, system (2.1) also has a 2π-periodic solution if rank $\| F_{sj}^*(\varepsilon) \| = m$, and this condition is verified by Taylor polynomials of order ν for the functions $F_{sj}^*(\varepsilon)$. Here the initial conditions for periodic solutions when $\mu = 0$ and $\mu \neq 0$ differ by a quantity of the order of $\varepsilon^{\nu+\sigma}$.

The case rank $\| F_{sj}^*(0) \| = m - k$ arises, in particular, when the solution $\mathbf{x} = \varphi(0, t)$ belongs to the k-family, i.e. system (2.1) when $\varepsilon = \mu = 0$ allows of a k-family of 2π-periodic solutions. In this case the functions $F_{\alpha j}$ ($\alpha = 1, \ldots, k$) in (2.3) vanish together with ε. Hence, system (2.3) is solvable for the weaker condition $\mu = O(\varepsilon^{1+\sigma})$, $\sigma > 0$.

Theorem 2. Suppose a 2π-periodic solution $\mathbf{x} = \varphi(\varepsilon, t)$ exists in the 2π-periodic system (2.1) when $\mu = 0$, which when $\varepsilon = 0$ belongs to a certain family. Then, in any class of functions $\mu = O(\varepsilon^{1+\sigma})$, $\sigma > 0$, system (2.1) also has a 2π-periodic solution, if rank $\| F^*_{sj}(\varepsilon) \| = m$, and this condition is verified taking into account terms in the matrix $\| F^*_{sj}(\varepsilon) \|$ that are in linear in ε. Here the initial conditions for periodic motions when $\mu = 0$ and $\mu \neq 0$ differ by a quantity of the order of ε^σ.

2. *The autonomous system*. In this case ΔT may be non-zero, and when analysing the compatibility of system (2.3) the matrix $\| F^*_{sj}(\varepsilon) \|$ must be replaced by the matrix $\| F^*_{sj}(\varepsilon), G^*_s(\varepsilon) \|$.

Theorem 3. Suppose a $2T^*$-periodic solution exists in autonomous system (2.1) when $\mu = 0$. Then, in any class of functions $\mu = O(\varepsilon^{2\nu+\sigma})$, $\sigma > 0$, system (2.1) also has a $2T(\varepsilon)$-periodic solution provided rank $\| F^*_{sj}(\varepsilon), G^*_s(\varepsilon) \| = m$, and this condition is verified by Taylor polynomials of order ν for the functions $F^*_{sj}(\varepsilon)$, $G^*_s(\varepsilon)$. In this case the initial conditions for periodic solutions when $\mu = 0$ and $\mu \neq 0$ differ by a quantity of the order of $\varepsilon^{\nu+\sigma}$ like their periods. If here rank $\| F^*_{sj}(\varepsilon) \| = m$, then when $\mu \neq 0$ in system (2.1) a $2T^*(\varepsilon)$-periodic solution also necessarily exists.

Similarly, as also in the periodic system, there is an important special case.

Theorem 4. Suppose a $2T^*$-periodic solution $\mathbf{x} = \varphi(\varepsilon, t)$ exists in the autonomous system (2.1) when $\mu = 0$, which belongs to a certain family when $\varepsilon = 0$. Then, in any class of functions $\mu = O(\varepsilon^{1+\sigma})$, $\sigma > 0$, system (2.1) also has a $2T(\varepsilon)$-periodic solution, provided that rank $\| F^*_{sj}(\varepsilon) \| = m$, and this condition is verified taking into account terms in the matrix $\| F^*_{sj}(\varepsilon), G^*_s(\varepsilon) \|$ that are linear in ε. In this case the initial conditions for periodic motions when $\mu = 0$ and $\mu \neq 0$ differ by a quantity of the order of $O(\varepsilon^\sigma)$, the same as their periods. If, in this case rank $\| F^*_{sj}(\varepsilon) \| = m$, then when $\mu \neq 0$ a $2T^*(\varepsilon)$-periodic solution also necessarily exist in the system.

Notes. 1. It is obvious that the conditions of Theorems 1–4 guarantee the uniqueness of the periodic solution of system (2.1) when $\mu = 0$ for each $\varepsilon \neq 0$; the generating system obtained from (2.1) when $\mu = 0$ is rough. A Poincaré-isolated case occurs in system (2.1) with one small parameter μ.

2. In Theorems 1–4 one does not need to know the periodic solution of system (2.1) when $\mu = 0$; it is sufficient to construct the periodic solution with accuracy ε^ν.

3. Constructively, the conditions rank $\| F^*_{sj}(\varepsilon) \| = m$, rank $\| F^*_{sj}(\varepsilon), G^*_s(\varepsilon) \| = m$ are verified by calculating (with the necessary accuracy with respect to ε) the characteristic exponents of the system of equations in variations in the neighbourhood of the solution $\mathbf{x} = \varphi(\varepsilon, t)$ of system (2.1) when $\mu = 0$.

4. In the autonomous case (Theorems 2 and 4) it is assumed that the family is defined not only by an arbitrary constant added in the solution to time.

3. A SYSTEM OF STANDARD FORM

We will assume that the generating system

$$\mathbf{x}^{\cdot} = \mathbf{X}(\varepsilon, \mathbf{x}, t), \quad \mathbf{x} \in \mathbf{R}^m \tag{3.1}$$

obtained from (2.1) with $\mu = 0$, allows of a family of k parameters $A_1, \ldots, A_k$ ($k \leqslant m$) of periodic solutions when $\varepsilon = 0$, where rank $\| \partial x_s^\circ / \partial A_j \| = k$. We will choose as the new variables of the problem $y_\alpha = A_\alpha$, $z_\beta (\alpha = 1, \ldots, k;\ \beta = k+1, \ldots, m)$ so that the transformations $(x_\alpha, x_\beta) \to (y_\alpha, y_\beta)$ is non-degenerate. For example, we can retain the variables x_β as the variables z_β. Then, as a rule, we arrive at the problem of the periodic motions of a system of standard form

$$\begin{aligned} \mathbf{y}^{\cdot} &= \varepsilon \mathbf{Y}(\varepsilon, \mathbf{y}, \mathbf{z}, t) + \mu \mathbf{Y}_1(\varepsilon, \mu, \mathbf{y}, \mathbf{z}, t) \\ \mathbf{z}^{\cdot} &= \mathbf{Z}_0(\mathbf{y}, \mathbf{z}, t) + \varepsilon \mathbf{Z}(\varepsilon, \mathbf{y}, \mathbf{z}, t) + \mu \mathbf{Z}_1(\varepsilon, \mu, \mathbf{y}, \mathbf{z}, t) \end{aligned} \tag{3.2}$$

which is of independent interest.

Note that in the case of autonomous system (2.1), among the parameters $A_1, \ldots, A_k$ there is no arbitrary constant added in the solution to t, and in the case when the period depends on $A_1, \ldots, A_k$, by introducing a new independent variable we arrive at a periodic system of the form (3.2) with a period which is independent of these constants.

We will investigate the following system, which is more general than (3.2)

$$y_\alpha^\bullet = \varepsilon^{p_\alpha} Y_\alpha(\varepsilon, \mathbf{y}, \mathbf{z}, t) + \mu_\alpha(\mu) Y_{1\alpha}(\varepsilon, \mu, \mathbf{y}, \mathbf{z}, t) \quad (\alpha = 1, \ldots, k)$$
$$z_\beta^\bullet = Z_{0\beta}(\mathbf{y}, \mathbf{z}, t) + \varepsilon Z_\beta(\varepsilon, \mathbf{y}, \mathbf{z}, t) + \mu Z_{1\beta}(\varepsilon, \mu, \mathbf{y}, \mathbf{z}, t) \quad (\beta = k+1, \ldots, m) \tag{3.3}$$

($p_\alpha \in \mathbb{N}$, $\mu_\alpha(0) = 0$, $\alpha = 1, \ldots, k$) with right-hand sides that are 2π-periodic in t, assuming that the system

$$z_\beta^\bullet = Z_{0\beta}(\mathbf{A}, \mathbf{z}, t), \quad \mathbf{A} = \text{const} \quad (\beta = k+1, \ldots, m) \tag{3.4}$$

allows of a 2π-periodic solution $\mathbf{z} = \varphi(\mathbf{A}, t)$.

The necessary and sufficient conditions for the solution of system (3.3) to be 2π-periodic have the form

$$y_\alpha(\varepsilon, \mu, \mathbf{y}^\circ, \mathbf{z}^\circ, 2\pi) - y_\alpha^\circ = 0 \quad (\alpha = 1, \ldots, k)$$
$$z_\beta(\varepsilon, \mu, \mathbf{y}^\circ, \mathbf{z}^\circ, 2\pi) - z_\beta^\circ = 0 \quad (\beta = k+1, \ldots, m) \tag{3.5}$$

($\mathbf{y}^\circ$, $\mathbf{z}^\circ$ are the initial conditions for $\mathbf{y}$ and $\mathbf{z}$, respectively), where the first group of equations in (3.5) is satisfied identically for $\mathbf{y}^\circ$, $\mathbf{z}^\circ$ when $\varepsilon = 0$, $\mu = 0$. Hence, taking into account the proportionality of the rate of change of the variable y_α to the quantity ε^{p_α}, we can write system (3.5) in the form

$$\xi_\alpha(\mathbf{y}^\circ, \mathbf{z}^\circ) + \xi_{1\alpha}(\varepsilon, \mathbf{y}^\circ, \mathbf{z}^\circ) + \mu_\alpha \varepsilon^{-p_\alpha} f_\alpha(\varepsilon, \mu, \mathbf{y}^\circ, \mathbf{z}^\circ) = 0 \quad (\alpha = 1, \ldots, k)$$
$$\eta_\beta(\mathbf{y}^\circ, \mathbf{z}^\circ) + \eta_{1\beta}(\varepsilon, \mathbf{y}^\circ, \mathbf{z}^\circ) + \mu g_\alpha(\varepsilon, \mu, \mathbf{y}^\circ, \mathbf{z}^\circ) = 0 \quad (\beta = k+1, \ldots, m) \tag{3.6}$$

where the functions $\xi_{1\alpha}(\varepsilon, \mathbf{y}^\circ, \mathbf{z}^\circ)$, $\eta_{1\beta}(\varepsilon, \mathbf{y}^\circ, \mathbf{z}^\circ)$ vanish when $\varepsilon = 0$. Hence it follows that by choosing $\mu_\alpha = o(\varepsilon^{p_\alpha})$, $\mu = o(\varepsilon)$ as $\varepsilon \to 0$, system (3.6) is compatible for sufficiently small $\varepsilon \neq 0$, if solutions of the system of equations

$$\xi_\alpha(\mathbf{y}^\circ, \mathbf{z}^\circ) = 0, \quad \eta_\beta(\mathbf{y}^\circ, \mathbf{z}^\circ) = 0 \quad (\alpha = 1, \ldots, k; \beta = k+1, \ldots, m) \tag{3.7}$$

exist and are simple.

The second group of equations in (3.7) for any $\mathbf{y}^\circ = \mathbf{A}$ allows of a solution $\mathbf{z}^\circ = \varphi(\mathbf{A}, 0)$. Hence, the problem of finding the roots of Eqs (3.7) leads to the compatibility of the system

$$\xi_\alpha(\mathbf{A}, \varphi(\mathbf{A}, 0)) = 0 \quad (\alpha = 1, \ldots, k) \tag{3.8}$$

The functions ξ_α in (3.8) are determined by integrating the system of differential equations

$$\xi_\alpha^\bullet = Y_\alpha(0, \mathbf{A}, \varphi(\mathbf{A}, t), t) \quad (\alpha = 1, \ldots, k)$$

in the interval $[0, 2\pi]$. Consequently, the roots of Eqs (3.8) are calculated from the system of amplitude equations

$$I_\alpha(\mathbf{A}) = \int_0^{2\pi} Y_\alpha(0, \mathbf{A}, \varphi(\mathbf{A}, t), t)\,dt = 0 \quad (\alpha = 1, \ldots, k) \tag{3.9}$$

A unique solution of (3.6) corresponds to each simple root $\mathbf{A}^*$ of these equations, which satisfy the condition $\det\| \partial\eta_\beta/\partial y_j^\circ \| \neq 0$ when $\mathbf{y}^\circ = \mathbf{A}^*$, $\mathbf{z}^\circ = \varphi(\mathbf{A}^*, 0)$ for sufficiently small $\varepsilon \neq 0$. We have thereby proved the existence of 2π-periodic motions in system (3.3). These motions are described by the formulae

$$y_\alpha = A_\alpha^* + \varepsilon^{p_\alpha} \int_0^t Y_\alpha(0, \mathbf{A}^*, \varphi(\mathbf{A}^*, t), t)\,dt + o(\varepsilon^{p_\alpha}) \quad (\alpha = 1, \ldots, k)$$
$$z_\beta = \varphi_\beta(\mathbf{A}^*, t) + O(\varepsilon) \quad (\beta = k+1, \ldots, m) \tag{3.10}$$

Theorem 5. To each simple root $\mathbf{A}^*$ of the amplitude equation (3.9), for which equations in variations for system (3.4) when $\mathbf{A} = \mathbf{A}^*$ do not have roots of the characteristic equation equal to unity, there corresponds a unique 2π-periodic solution of system (3.3), if $\mu_\alpha = o(\varepsilon^{p_\alpha})$, $\mu = o(\varepsilon)$ and $\varepsilon \to 0$.

Notes. 1. When the right-hand sides of system (3.3) are 2π-periodic in the variable $z_{l+1}, \dots, z_m (l \geq k)$, Theorem 5 also establishes the existence of rotational motions, 2π-periodic on $\mathbb{R} \times \mathbb{T}^{m-l}$ ($\mathbb{T}^{m-l}$ is a torus of dimension $m-l$). For such motions the solution $\mathbf{z} = \varphi(\mathbf{A}, t)$ satisfies the conditions

$$\varphi_i(\mathbf{A}, t+2\pi) = \varphi_i(\mathbf{A}, t), \quad \varphi_j(\mathbf{A}, t+2\pi) - \varphi_j(\mathbf{A}, t) = 0 (\mathrm{mod}\, 2\pi)$$
$$(i = k+1, \dots, l; j = l+1, \dots m)$$

2. Theorem 5 allows of an extension to the case when Eq. (3.4), for fixed values p of the parameters $A_1 = A_1^*, \dots, A_p = A_p^*$ and arbitrary values of the remaining parameters $A_{p+1}, \dots, A_k$, allows of a family of $B_1, \dots, B_p$ 2π-periodic solutions $\mathbf{z} = \varphi(A_1^*, \dots, A_p^*, A_{p+1}, \dots, A_k, B_1, \dots, B_p, t)$.

Corollary. To each simple root $\mathbf{A}^*$ of the amplitude equation

$$\mathbf{I}(\mathbf{A}) \equiv \int_0^{2\pi} \mathbf{Y}(0, \mathbf{A}, t) dt = 0 \tag{3.11}$$

of the 2π-periodic system

$$\mathbf{y}^{\cdot} = \mu \mathbf{Y}(\mu, \mathbf{y}, t), \quad \mathbf{y} \in \mathbb{R}^m \tag{3.12}$$

there corresponds, for sufficient small $\mu \neq 0$ a unique 2π-periodic solution

$$\mathbf{y} = \mathbf{A}^* + \mu \int_0^t \mathbf{Y}(0, \mathbf{A}\ , t) dt + o(\mu)$$

of system (3.12).

Example. A quasilinear system with one degree of freedom.

1. The autonomous system

$$x^{\cdot\cdot} + \omega^2 x = \mu F(\mu, x, x^{\cdot}); \quad \omega = \mathrm{const} > 0 \tag{3.13}$$

When $\mu = 0$ we have a family of $2\pi/\omega$-periodic with respect to t solutions $x = A\cos\varphi$ that is single-parametric from A. We change to the Van der Pol variables A and φ. As a result we obtain a system which we can write in the form of a single equation

$$\frac{dA}{d\varphi} = -\frac{\mu}{\omega} \frac{F(\mu, A\cos\varphi, -A\omega\sin\varphi)\sin\varphi}{\omega - \mu(\omega A)^{-1} F(\mu, A\cos\varphi, -A\omega\sin\varphi)\cos\varphi}$$

with a right-hand side that is 2π-periodic in φ. From the corollary of Theorem 5 we derive the following [2]: to each simple root A^* of the amplitude equation

$$\int_0^{2\pi} F(0, A\cos\varphi, -A\omega\sin\varphi)\sin\varphi d\varphi = 0$$

there corresponds a unique solution, 2π-periodic in φ

$$A = A^* + \mu \int_0^{\varphi} F(0,\ A\cos\varphi,\ -A\omega\sin\varphi)\sin\varphi d\varphi + o(\mu)$$

$$\frac{d\varphi}{dt} = \omega - \frac{\mu}{\omega A^*} \int_0^{\lambda} F(0, A\cos\varphi,\ -A\omega\sin\varphi)\cos\varphi d\varphi + o(\mu)$$

of Eq. (3.13).

2. A periodic system with a principal resonance

$$x^{\cdot\cdot} + \omega^2 x = \mu F(\mu, x, x^{\cdot}, t), \quad F(\mu, x, x^{\cdot}, t+2\pi) = F(\mu, x, x^{\cdot}, t), \quad \omega^2 = 1 - \mu a \tag{3.14}$$

When $\mu = 0$, Eq. (3.14) allows of a family of 2π-periodic solutions, which is two-parametric from A and B

$$x = A\cos t + B\sin t, \ x^{\cdot} = -A\sin t + B\cos t$$

We make the replacement $(x, x^*) \rightarrow (A, B)$ in (3.12). As a result we obtain

$$A^{\bullet}=-\mu F^{*}(\mu,A,B,t)\sin t,\quad B^{\bullet}=\mu F^{*}(\mu,A,B,t)\cos t$$
$$F^{*}(\mu,A,B,t)\equiv a+F(\mu,A\cos t+B\sin t,\ -A\sin t+B\cos t,t)$$

According to the corollary of Theorem 5, to each simple root (A^*, B^*) of the amplitude equation

$$P(A,B)\equiv\int_0^{2\pi}F^{*}(0,A,B,t)\sin t dt=0,\quad Q(A,B)\equiv\int_0^{2\pi}F^{*}(0,A,B,t)\cos t dt=0$$
$$\partial(P,Q)/\partial(A,B)|_{(A^*,B^*)}\neq 0$$

there corresponds a unique 2π-periodic solution

$$x=\left[A^{*}-\mu\int_0^t F^{*}(0,A^{*},B^{*},t)\sin t dt\right]\cos t+\left[B^{*}+\mu\int_0^t F^{*}(0,A^{*},B^{*},t)\cos t dt\right]\sin t+o(\mu)$$

of system (3.14).

4. DEGENERATE SYSTEMS

1. *A resonant system.* The problem of the rotational motions of a system in the resonant case also leads to an investigation of system (3.12).

Consider the following system, 2π-periodic in t

$$\begin{aligned}&y_\alpha^{\bullet}=\mu Y_\alpha(\mu,\mathbf{y},\mathbf{z},t)\quad(\alpha=1,\dots,k)\\&z_\beta^{\bullet}=\omega_\beta+\mu Z_\beta(\mu,\mathbf{y},\mathbf{z},t),\quad \omega_\beta=\text{const}\quad(\beta=k+1,\dots,m)\end{aligned}\tag{4.1}$$

in which the right-hand sides are also 2π-periodic with respect to the variable z_β. We will assume that system (4.1) is resonant, i.e. $\omega_\beta=p_\beta/q_\beta-a_\beta\mu$ ($\beta=k+1,\dots,m$), where $p_\beta\, q_\beta\in\mathbb{Z}(q_\beta\neq 0)$, $a_\beta=$ const. In this case, by introducing the new variables $\zeta_\beta=z_\beta-(p_\beta/q_\beta)t$ ($\beta=k+1,\dots,m$) we obtain the following system, $2\pi l$-periodic in t (l is the least common multiple of the numbers $|q_{k+1}|,\dots,|q_m|$)

$$\begin{aligned}&y_\beta^{\bullet}=\mu Y_\beta(\mu,y_1,\dots,y_k,\ \zeta_{k+1}+(p_{k+1}/q_{k+1})t,\dots,\zeta_m+(p_m/q_m)t,t)\\&\zeta_\beta^{\bullet}=\mu[a_\beta+Z_\beta(\mu,y_1,\dots,y_k,\zeta_{k+1}+(p_{k+1}/q_{k+1})t,\dots,\zeta_m+(p_m/q_m)t,t)]\end{aligned}\tag{4.2}$$
$$(\alpha=1,\dots,k;\beta=k+1,\dots,m)$$

of the form (3.12). According to the corollary of Theorem 5, to each simple root $(\mathbf{A}^*, \mathbf{B}^*)$ of the amplitude equation

$$\int_0^{2\pi l}\mathbf{Y}(0,\mathbf{A},\mathbf{B}+(\mathbf{p}/\mathbf{q})t,t)dt=\mathbf{0},\quad\int_0^{2\pi l}[\mathbf{a}+\mathbf{Z}(0,\mathbf{A},\mathbf{B}+(\mathbf{p}/\mathbf{q})t,t)dt=\mathbf{0}$$

there corresponds a unique $2\pi l$-periodic solution of system (4.2). The corresponding solution of system (4.1) has the form

$$y_\alpha=A_\alpha^{*}+\mu\int_0^t Y_\alpha(0,\mathbf{A}^{*},\mathbf{B}^{*}+(\mathbf{p}/\mathbf{q})t,t)dt+o(\mu)\quad(\alpha=1,\dots,k)$$

$$z_\beta=\omega_\beta t+B_\beta^{*}+\mu\int_0^t Z_\beta(0,\mathbf{A}^{*},\mathbf{B}^{*}+(\mathbf{p}/\mathbf{q})t,t)dt+o(\mu)\quad(\beta=k+1,\dots,m)$$

2. *The case of multiple roots of the amplitude equation.* We will assume that in the $2\pi l$-periodic system (3.12) the amplitude equation (3.11) has a multiple root $\mathbf{A}^*$, where

$$\operatorname{rank}\left\|\frac{\partial\mathbf{I}(\mathbf{A})}{\partial\mathbf{A}}\right\|_{\mathbf{A}^*}=m-k\tag{4.3}$$

and simple elementary dividers correspond to the root $\mathbf{A}^*$ in (4.3). In this case, we make the following replacement in system (3.12)

$$\mathbf{y} = \mathbf{A}^* + \mu^\sigma \mathbf{z} + \mu \mathbf{f}(t) \quad (0 < \sigma \leq 1), \quad \mathbf{f}(t) = \int_0^t \mathbf{Y}(0, \mathbf{A}^*, t) dt$$

As a result we obtain

$$\mathbf{z}^\cdot = \mu\{\mathbf{P}_1(t)[\mathbf{z} + \mu^{1-\sigma}\mathbf{f}(t)] + \mu^\sigma \mathbf{P}_2(t)[\mathbf{z} + \mu^{1-\sigma}\mathbf{f}(t)]^2 + \mu^{1-\sigma}\partial \mathbf{Y}/\partial\mu + \ldots\}$$
$$\mathbf{P}_1(t) = \partial \mathbf{Y}(0, \mathbf{A}, t)/\partial \mathbf{A}, \quad 2\mathbf{P}_2(t) = \partial^2 \mathbf{Y}(0, \mathbf{A}, t)/\partial \mathbf{A}^2$$

where all the partial derivatives are evaluated with $\mu = 0$, $\mathbf{A} = \mathbf{A}^*$.

We will first assume that the matrix $\mathbf{P}_1(t)$ is constant. In this case condition (4.3) and the simple elementary dividers in (4.3) guarantee the existence of a non-degenerate linear transformation $\mathbf{z} \to (\xi, \eta)$ with constant coefficients, such that, in the new variables, the system takes the form

$$\begin{aligned} &\xi_\alpha^\cdot = \mu^{1+\sigma}[\Xi_{0\alpha}(\xi, \eta, t) + \mu^\sigma \Xi_{1\alpha}(\mu, \xi, \eta, t) + \mu^{1-2\sigma}\Xi_{0\alpha}^*(\xi, \eta, t) + \mu^{1-\sigma}\Xi_{1\alpha}^*(\mu, \xi, \eta, t)] \\ &\eta_\beta^\cdot = \mu\left[\sum_{j=k+1}^m Q_{\beta j}\eta_j + \mu^\sigma H_{0\beta}(\mu, \xi, \eta, t) + \mu^{1-2\sigma} H_{1\beta}(\mu, \xi, \eta, t)\right] \\ &(\alpha = 1, \ldots, k; \beta = k+1, \ldots, m) \end{aligned} \tag{4.4}$$

where the linear system $\eta^* = \mathbf{Q}\eta$ has a unique (zero) 2π-periodic solution.

Obviously, (4.4) is a system of the form (3.3) with $p_\alpha = 1 + \sigma$, $p_\beta = 1$.

Consider the system of k equations

$$\int_0^{2\pi} [\Xi_{0\alpha}(\mathbf{B}, 0, t) + \Xi_{0\alpha}^*(\mathbf{B}, 0, t)] dt = 0 \quad (\alpha = 1, \ldots, k) \tag{4.5}$$

in the k constants $B_1, \ldots, B_k$. Suppose $\mathbf{B}^*$ is a simple Broot of this system. Then, we put $\sigma = 1/2$ in (4.4) and, using Theorem 5, we derive the existence in system (4.4) of a unique 2π-periodic solution

$$\xi_\alpha = B_\alpha^* + \mu^{3/2}\int_0^t [\Xi_{0\alpha}(\mathbf{B}^*, 0, t) + \Xi_{0\alpha}^*(\mathbf{B}^*, 0, t)] dt + o(\mu^{3/2}), \quad \eta_\beta = o(\mu)$$

with $\mu \neq 0$. Consequently, system (3.12), for fairly small $\mu \neq 0$, has a unique 2π-periodic solution, which differs from the generating constant solution $\mathbf{A}^*$ by a quantity of the order of $\mu^{1/2}$.

The case of multiple roots of the system of equations (4.5) is reduced by the above scheme to an analysis of a new system of amplitude equations. In the case when the integrals in (4.5) are identically zero, it is necessary to analyse the second derivatives of the function $\mathbf{Y}(\mu, \mathbf{y}, t)$ with respect to μ and to choose $\sigma^{1/3}$. As a result, we establish the existence of a periodic solution in $O(\mu^{1/3})$—the neighbourhood of the generating solutions $\mathbf{A}^*$. Other degenerate cases can be investigated similarly (see also [5–7]).

In the case of a periodic matrix $\mathbf{P}_1(t)$, the above scheme does not undergo any appreciable changes if the linear system $\mathbf{z}^\cdot = \mathbf{P}_1(t)\mathbf{z}$ has a k-family of 2π-periodic solutions. This occurs, for example, when $m = k = 1$.

5. PERIODIC SOLUTIONS IN THE μ^σ-NEIGHBOURHOOD OF THE GENERATING SOLUTION (THE GENERAL CASE)

In Sections 2–4 we considered limiting cases when the property of the system has a periodic solution, and is determined solely by the generating system (Section 2) or only by the perturbations (Sections 3 and 4). At the same time, an investigation of the case of multiple roots of the amplitude equation (Section 4, paragraph 2) shows that in the general case all terms on the right-hand side of (1.1) correspond to the existence of a periodic solution. Another important feature is the fact that, when investigating non-rough cases, the fact that the periodic solution $\mathbf{x} = \varphi(t)$ of the generating system (1.2) belongs to a certain family is not always known in advance. These facts enable us to consider the following system instead of Eqs (1.1) in the general case

$$\mathbf{y}^{\cdot} = \mathbf{P}(t)\mathbf{y} + \mathbf{Y}(\mathbf{y}, t) + \mu\mathbf{X}_1(\mu, \mathbf{w}(t) + \mathbf{y}, t), \quad \mathbf{y} \in \mathbf{R}^m \tag{5.1}$$

obtained by making the change $\mathbf{y} = \mathbf{x} - \varphi(t)$ in the neighbourhood of the periodic solution $\varphi(t)$.

We will assume that the matrix $\mathbf{P}(t)$ and the functions $\mathbf{Y}$ and $\mathbf{X}_1$ are 2π-periodic in t, while the linear system

$$\mathbf{y}^{\cdot} = \mathbf{P}(t)\mathbf{y} \tag{5.2}$$

allows of a k-parametric family from $B_1, \ldots, B_k$

$$\mathbf{y} = B_1\boldsymbol{\theta}_1(t) + \ldots + B_k\boldsymbol{\theta}_k(t) \equiv \varphi(\mathbf{B}, t) \tag{5.3}$$

of 2π-periodic solutions. We thereby encompass both the case of a family of periodic solutions of generating system (1.2), and also the case when we know that the solutions $\mathbf{x} = \varphi(t)$ belong to a certain family.

Note that the condition for the family (5.3) to exist is satisfied when there are exactly k simple zero characteristic exponents, and the latter condition is constructively verified. This case is investigated below.

We will make the replacement $\mathbf{y} \to \varepsilon\mathbf{y}$ of an already small parameter $\varepsilon = \mu^\sigma$ $(0 < \sigma \leqslant 1)$, characterizing the smallness of the deviation of the solutions of systems (1.1) and (1.2) from one another. We obtain as a result

$$\begin{gathered}\mathbf{y}^{\cdot} = \mathbf{P}(t)\mathbf{y} + \varepsilon^\lambda[\mathbf{Y}_0(\mathbf{y}, t) + \varepsilon\mathbf{Y}_1(\varepsilon, \mathbf{y}, t)] + \mu^{1-\sigma}[\mathbf{X}_1(0, \varphi(t), t) + \varepsilon\mathbf{Q}(t)\mathbf{y} + \\ + \mu\mathbf{R}(t) + \mathbf{X}_2(\mu, \varepsilon, \mathbf{y}, t)]; \quad \mathbf{Q}(t) = \left(\frac{\partial\mathbf{X}_1(\mu, \mathbf{x}, t)}{\partial\mathbf{x}}\right)_*, \quad \mathbf{R}(t) = \left(\frac{\partial\mathbf{X}_1(\mu, \mathbf{x}, t)}{\partial\mu}\right)_*\end{gathered} \tag{5.4}$$

where $\lambda \geqslant 1$ $(\lambda \in \mathbb{N})$, the matrix $\mathbf{Q}(t)$ and the vector $\mathbf{R}(t)$ are calculated for $\mathbf{x} = \varphi(t)$, $\mu = 0$, while the function $\mathbf{X}_2$ $(\mu, \varepsilon, \mathbf{y}, t)$ vanishes when $\varepsilon = \mu = 0$ and is an order higher than the first in ε and μ.

We will denote by $\{z_{s\alpha}(t)\}$ the system of 2π-periodic solutions of the linear system, conjugate with system (5.2), and by making the replacement $\mathbf{y} \to (\boldsymbol{\xi}, \boldsymbol{\eta})$ we reduce (5.2) to a system with constant coefficients. Then

$$\begin{gathered}\xi_\alpha^{\cdot} = \sum_{s=1}^m \left\{\varepsilon^\lambda(Y_{0s} + \varepsilon Y_{1s}) + \mu^{1-\sigma}\left[X_{1s}(0, \varphi(t), t) + \varepsilon\sum_{j=1}^m Q_{sj}y_j + \mu R_s + X_{2s}\right]\right\} z_{s\alpha}(t) \\ \boldsymbol{\eta}^{\cdot} = \mathbf{G}\boldsymbol{\eta} + \mathbf{H}(\varepsilon, \mu, \boldsymbol{\xi}, \boldsymbol{\eta}, t) \quad (\alpha = 1, \ldots, k; \ \boldsymbol{\eta} \in \mathbf{R}^{m-k})\end{gathered} \tag{5.5}$$

and among the eigenvalues of the matrix $\mathbf{G}$ there is none equal to il $(l \in \mathbb{Z})$.

We will assume initially that the following conditions are satisfied on the solution $\varphi(t)$

$$\int_0^{2\pi} \sum_{s=1}^m X_{1s}(0, \varphi(t), t) z_{s\alpha}(t)dt = 0 \quad (\alpha = 1, \ldots, k) \tag{5.6}$$

which are necessary for a 2π-periodic solution to exist in system (1.1), apart from terms linear in μ. Then, by the transformation

$$\xi_\alpha \to \xi_\alpha + \mu^{1-\sigma}\int_0^t \sum_{s=1}^m X_{1s}(0, \varphi(t), t) z_{s\alpha}(t)dt$$

we obtain a system of the form (5.5), in which X_{1s} $(0, \varphi(t), t) \equiv 0$. This system is of the type (3.3), and the problem of the existence of a periodic solution in it reduces to analysing an amplitude equation.

Consider the following system of amplitude equations

$$I_\alpha(\mathbf{B}) \equiv \int_0^{2\pi} \sum_{s=1}^m \left[\sum_{j=1}^m Q_{sj}(t)\psi_j(\mathbf{B}, t) z_{s\alpha}(t)\right] dt = 0 \quad (\alpha = 1, \ldots, k) \tag{5.7}$$

$$I_\alpha(\mathbf{B}) + \int_0^{2\pi} \sum_{s=1}^m Y_{0s}(\boldsymbol{\psi}(\mathbf{B}, t), t) z_{s\alpha}(t)dt = 0 \quad (\alpha = 1, \ldots, k) \tag{5.8}$$

$$I_\alpha(\mathbf{B})+\int_0^{2\pi}\sum_{s=1}^{m} R_s(t)z_{s\alpha}(t)dt=0 \quad (\alpha=1,\ldots,k) \tag{5.9}$$

$$I_\alpha(\mathbf{B})+\int_0^{2\pi}\sum_{s=1}^{m} [Y_{0s}(\psi(\mathbf{B}),t)+R_s(t)]z_{s\alpha}(t)dt=0 \quad (\alpha=1,\ldots,k) \tag{5.10}$$

System (5.7) is a system of linear algebraic equations and, for a non-zero determinant Δ, has a unique (zero) solution. In this case, assuming $1/\lambda < \sigma < 1$ in (5.5), we arrive, on the basis of Theorem 5, at the existence in (1.1) when $\lambda > 1$ of a unique 2π-periodic solution

$$\mathbf{x}=\varphi(t)+\mu\varphi_1(t)+o(\mu) \tag{5.11}$$

If we know that, in the generating system (1.2), the solution $\varphi(t)$ belongs to the family $\varphi(\mathbf{A}, t)$ of parameters $A_1, \ldots, A_k$, Eqs (5.6) serve to determine the quantities $\mathbf{A} = \mathbf{A}^*$ in the generating solution. The sufficient condition for the solution to be continued over μ is $\Delta \neq 0$.

When a simple root $\mathbf{B}^*$ of system (5.8) exists, we put $\sigma = 1/\lambda$. Then, on the basis of Theorem 5, we conclude that the following 2π-periodic solution exists in (1.1) when $\lambda > 1$

$$\mathbf{x}=\varphi(t)+\mu^\sigma\varphi(\mathbf{B}^*,t)+o(\mu^\sigma), \quad \sigma=1/\lambda \tag{5.12}$$

Obviously such a situation is possible both when $\Delta \neq 0$ and when $\Delta = 0$.

When $\Delta \neq 0$, Eqs (5.9) also have a unique, in the general case, non-zero solution $\mathbf{B}^*$. When $\lambda > 1$ we choose $\sigma = 1$, and on the basis of Theorem 5, we conclude that the following 2π-periodic solution exists in (1.1)

$$\mathbf{x}=\varphi(t)+\mu[\varphi_1(t)+\psi(\mathbf{B}^*,t)]+o(\mu) \tag{5.13}$$

When $\lambda = 1$ the periodic solution also has the form (5.13), if system (5.10) has the simple solution $\mathbf{B}^*$. This is possible both when $\Delta \neq 0$ and when $\Delta = 0$. When the solution $\varphi(t)$ in the generating system belongs to the family $\varphi(\mathbf{A}, t)$ of parameters $A_1, \ldots, A_k$, Eq. (5.10) acquires the form (5.9), and the sufficient condition for the periodic solution to be continued over μ is $\Delta \neq 0$.

We will now assume that conditions (5.6) are not satisfied. In this case we put $\sigma = 1/(1 + \lambda)$, and on the basis of Theorem 5 we conclude that the periodic solution

$$\mathbf{x}=\varphi(t)+\mu^\sigma\psi(\mathbf{B}^*,t)+o(\mu^\sigma), \quad \sigma=1/(1+\lambda) \tag{5.14}$$

corresponding to the simple root $\mathbf{B}^*$ of the amplitude equation

$$\int_0^{2\pi}\sum_{i=1}^{m} [Y_{0s}(\psi(\mathbf{B},t),t)+X_{1s}(0,\varphi(t),t]z_{s\alpha}(t)dt=0 \quad (\alpha=1,\ldots,k) \tag{5.15}$$

exists.

Theorem 6. A single 2π-periodic solution of system (1.1) corresponds to each simple root $\mathbf{B}^*$ of any of the systems of amplitude equations (5.7)–(5.10) and (5.15) provided μ is sufficiently small. Here the periodic solutions in cases (5.7)–(5.10) exist when conditions (5.6) are satisfied and have the form (5.11)–(5.13), respectively, a solution of the form (5.13) corresponds to cases (5.9) and (5.10), and the generation of all three types (5.11)–(5.13) of periodic solutions simultaneously is possible. The periodic solution in the case of (5.15) has the form (5.14), exists, when conditions (5.6) are not satisfied, and differs from the periodic solution of the generating system by a quantity of the order of μ^σ ($\sigma = 1/(1 + \lambda)$).

6. AN INVERSE SYSTEM

The theory of periodic motions for an inverse system was developed in [8–14]. We develop this theory further below.

1. Motion on a fixed set. The following assertion holds.

Theorem 7. Any motion of a 2π-periodic inverse system

$$\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u},t), \quad \mathbf{U}(\mathbf{u},t+2\pi) = \mathbf{U}(\mathbf{u},t), \quad \mathbf{U}(\mathbf{u},-t) = -\mathbf{U}(\mathbf{u},t); \quad \mathbf{u} \in \mathbf{R}^l \tag{6.1}$$

is 2π-periodic or a position of equilibrium.

In fact, each solution $\mathbf{u}_1 = \varphi(\mathbf{u}^\circ, t)$ is determined by the value $\mathbf{u}^\circ$ of the variable $\mathbf{u}$ at the instant of time $t = 0$. The property of invertibility also guarantees the existence, together with the solution $\varphi(\mathbf{u}^\circ, t)$, of the solution $\mathbf{u}_2 = \varphi(\mathbf{u}^\circ, -t)$. The existence of one other solution $\mathbf{u}_3 = \varphi(\mathbf{u}^\circ, 2\pi - t)$ follows from the 2π-periodicity of system (6.1). Further, we have

$$\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}^\circ, \quad \mathbf{u}_1(\pi) = \mathbf{u}_3(\pi) = \varphi(\mathbf{u}^\circ, \pi)$$

Consequently, $\mathbf{u}_1(t)$, $\mathbf{u}_2(t)$ and $\mathbf{u}_3(t)$ describe one and the same solution. This solution will be 2π-periodic because $\mathbf{u}_1(2\pi) = \mathbf{u}_3(0) = \mathbf{u}^\circ$. In a special case it degenerates into the equilibrium position $\mathbf{u}^*$, found from the equation $\mathbf{U}(\mathbf{u}^*, t) = \mathbf{0}$.

Note. All solutions of autonomous inverse system (6.1) are equilibrium positions.

Corollaries. 1. In the inverse 2π-periodic system

$$\mathbf{u}^{\cdot} = \mu\mathbf{U}(\mu,\mathbf{u},t), \quad \mathbf{U}(\mu,\mathbf{u},-t) = -\mathbf{U}(\mu,\mathbf{u},t); \quad \mathbf{u} \in \mathbf{R}^l$$

with small parameter μ, when $\mu \neq 0$ a 2π-periodic motion is generated from each equilibrium position if the equation $\mathbf{U}(\mu, \mathbf{u}^*, t)$ has no constant solutions $\mathbf{u}^*$.

2. All motions of the autonomous or 2π-periodic inverse system

$$\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u},\mathbf{v},t), \quad \mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{u},\mathbf{v},t); \quad \mathbf{u} \in \mathbf{R}^l, \quad \mathbf{v} \in \mathbf{R}^n$$

which belong, as a whole, to the fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$, are 2π-periodic or positions of equilibrium.

3. The linear 2π-periodic system

$$\mathrm{x}^{\cdot} = \mathrm{P}(t)\mathrm{x}, \quad \mathrm{P}(-t) = -\mathrm{P}(t); \quad \mathrm{x} \in \mathbf{R}^m$$

is stable and has only 2π-periodic or constant solutions.

2. *Cesari's method*. Consider the 2π-periodic inverse system

$$\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u},\mathbf{v},t) + \mu\mathbf{U}_1(\mu,\mathbf{u},\mathbf{v},t), \quad \mathbf{v}^{\cdot} = \mu\mathbf{V}_1(\mu,\mathbf{u},\mathbf{v},t); \quad \mathbf{u} \in \mathbf{R}^l, \quad \mathbf{v} \in \mathbf{R}^n \tag{6.2}$$

with fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$ and small parameter μ. When $\mu = 0$ system (6.2) allows of an l-family of 2π-periodic motions

$$\mathbf{u} = \varphi(\mathbf{A},t), \quad \mathbf{v} = \mathbf{0}, \quad \varphi^{\cdot} = \mathbf{U}(\varphi,\mathbf{0},t) \tag{6.3}$$

which belong to a fixed set (Theorem 7). We will investigate the question of whether a 2π-periodic motion exists in system (6.2) when $\mu \neq 0$, which changes into one of the motions (6.3) when $\mu = 0$.

Suppose $\mathbf{u}(\mu, \mathbf{u}^\circ, \mathbf{v}^\circ, t)$, $\mathbf{v}(\mu, \mathbf{u}^\circ, \mathbf{v}^\circ, t)$ is the general solution of system (6.2) with initial conditions $\mathbf{u}^\circ$, $\mathbf{v}^\circ$ when $t = 0$. Then, the necessary and sufficient conditions for $2\pi k$-periodicity of the solution, symmetrical about the fixed set, have the form

$$v_s(\mu, u_1^\circ, \ldots, u_l^\circ, 0, \ldots, 0, \pi k) = 0 \quad (s = 1, \ldots, n)$$

This system is satisfied identically when $\mu = 0$ and any $k \in \mathbb{N}$. Hence, we can represent system (6.3) in the form

$$f_s(\mathbf{u}^\circ, \pi k) + \mu g_s(\mu, \mathbf{u}^\circ, \pi k) = 0, \quad f_s(\mathbf{u}^\circ, \pi k) \equiv \left.\frac{\partial v_s(\mu, \mathbf{u}^\circ, t)}{\partial \mu}\right|_{\mu=0, t=\pi k} \tag{6.4}$$

($s = 1, \ldots, n$) and in the case of the root $\mathbf{u}^*$ of the system of equations $fs(\mathbf{u}^\circ, \pi k) = 0$ ($s = 1, \ldots, n$), which satisfy the condition rank $\| \partial f_s / \partial \mathbf{u}_j^* \| = n$, system (6.4) admits of a solution for sufficiently small

$\mu \neq 0$. The partial derivatives in (6.4) are found from the equations

$$\frac{d}{dt}\left(\frac{\partial v_s}{\partial \mu}\right) = v_{1s}(\mu, \mathbf{u}, \mathbf{v}, t) + \mu \frac{\partial}{\partial \mu} v_{1s}(\mu, \mathbf{u}, \mathbf{v}, t) \quad (s = 1, \ldots, n) \tag{6.5}$$

with zero initial conditions. Substituting the general solution of system (6.2) into the right-hand side of (6.5) and assuming $\mu = 0$, we obtain

$$\frac{\partial v_s(0, \mathbf{u}^\circ, \mathbf{0}, t)}{\partial \mu} = \int_0^{2\pi} v_{1s}(0, \mathbf{u}^\circ, \mathbf{0}, t)dt \quad (s = 1, \ldots, n) \tag{6.6}$$

Consequently, the following theorem holds.

Theorem 8. To each root $\mathbf{A}^*$ of the amplitude equation

$$\mathbf{I}(\mathbf{A}) \equiv \int_0^{\pi k} \mathbf{V}_1(0, \boldsymbol{\varphi}(\mathbf{A}, t), \mathbf{0}, t)dt = \mathbf{0}$$

for which rank $\| \partial \mathbf{I}(\mathbf{A}^*)/\partial \mathbf{A}^* \| = n$, there corresponds a $2\pi k$-periodic solution, symmetrical with respect to the fixed set $\mathbf{M}$

$$\mathbf{u} = \boldsymbol{\varphi}(\mathbf{A}^*, t) + \mu \int_0^t \left[\left(\frac{\partial \mathbf{U}}{\partial \mathbf{v}}\right)_* \boldsymbol{\psi}(t) + \mathbf{U}_1(0, \boldsymbol{\varphi}(\mathbf{A}^*, t))\right] dt + o(\mu)$$

$$\mathbf{v} = \mu \boldsymbol{\psi}(t) + o(\mu), \quad \boldsymbol{\psi}(t) = \int_0^t \mathbf{V}_1(0, \boldsymbol{\varphi}(\mathbf{A}^*, t), t)dt$$

(the asterisk denotes a calculation with $\mu = 0$, $\mathbf{u} = \boldsymbol{\varphi}(\mathbf{A}^*, t)$ of system (6.2).

Notes. 1. It is obvious that the conditions of Theorem 8 can only be satisfied when $l \geqslant n$. When $l > n$, Theorem 8 establishes the existence of an $(l - n)$-family of $2\pi k$-periodic motions.

2. Theorem 8 establishes the existence of periodic motions in the μ-neighbourhood of the fixed set.

3. In Cesari's method [15, 16] the case $\mathbf{U}(\mathbf{u}, \mathbf{v}, t) \equiv 0$ is considered.

3. A system of standard form. Consider the inverse $2\pi k$-periodic system

$$\mathbf{u}^\cdot = \mu \mathbf{U}_1(\mu, \mathbf{u}, \mathbf{v}, t), \quad \mathbf{v}^\cdot = \mathbf{V}(\mathbf{u}) + \mu \mathbf{V}_1(\mu, \mathbf{u}, \mathbf{v}, t); \quad \mathbf{u} \in \mathbb{R}^l, \quad \mathbf{v} \in \mathbb{R}^n \tag{6.7}$$

with fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$ and small parameter μ. We will assume that the right-hand sides are also 2π-periodic in $\mathbf{v}$.

When $\mu = 0$, system (6.7) allows of a family of solutions, symmetric about $\mathbf{M}$

$$\mathbf{u} = \mathbf{A}, \ \mathbf{v} = \mathbf{V}(\mathbf{A})t \tag{6.8}$$

Consider those of the solutions (6.8) which satisfy the conditions

$$v_s(\mathbf{A}) = p_s / q_s + \mu \alpha_s; \quad p_s \in \mathbb{Z}, \quad q_s \in \mathbb{N}, \quad \alpha_s = \text{const} \quad (s = 1, \ldots, n) \tag{6.9}$$

for which we make the replacement

$$u_j = A_j + \xi_j, \quad v_s = (p_s / q_s)t + \eta_s \quad (j = 1, \ldots, l; \ s = 1, \ldots, n)$$

As a result we obtain the system

$$\boldsymbol{\xi}^\cdot = \mu \mathbf{U}_1(\mu, \mathbf{A} + \boldsymbol{\xi}, (\mathbf{p}/\mathbf{q})t + \boldsymbol{\eta}, t) \tag{6.10}$$

$$\boldsymbol{\eta}^\cdot = \mathbf{P}(\mathbf{A})\boldsymbol{\xi} + \mathbf{H}(\mathbf{A}, \boldsymbol{\xi}) + \mu[\boldsymbol{\alpha} + \mathbf{V}_1(\mu, \mathbf{A} + \boldsymbol{\xi}, (\mathbf{p}/\mathbf{q})t + \boldsymbol{\eta}, t)$$

($\mathbf{H}$ are terms non-linear in $\boldsymbol{\xi}$) with right-hand sides that are $2\pi q$-periodic in t, where q is the least common multiple of the numbers $q_1, \ldots, q_n$.

Suppose rank $\mathbf{P} = k \leqslant n$. Then numbers $\varkappa_{1\nu}, \ldots \varkappa_{n\nu}$, $(\nu = k+1, \ldots, n)$ exist such that

$$\varkappa_{1\nu} p_{1j} + \ldots, + \varkappa_{n\nu} p_{nj} = 0 \quad (j = 1, \ldots, l; \;\; \nu = k+1, \ldots, n)$$

Hence, by means of a linear transformation with constant coefficients we can always reduce the last $n-k$ equations to a form in which, when $\mu = 0$, there are no terms linear in ξ. We now make the replacement $(\xi, \eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_n) \to (\mu^\sigma \xi, \mu^\sigma \eta_1, \ldots, \mu^\sigma \eta_k, \eta_{k+1}, \ldots, \eta_n)$, $1/2 < \sigma < 1$. Then, Eqs (5.10) take the form

$$\begin{aligned}
&\xi^{\cdot} = \mu^{1-\sigma} \mathbf{U}_1\left(\mu, \mathbf{A} + \mu^\sigma \xi, \frac{p_1}{q_1} t + \mu^\sigma \eta_1, \ldots, \frac{p_k}{q_k} t + \mu^\sigma \eta_k, \frac{p_{k+1}}{q_{k+1}} t + \eta_{k+1}, \ldots, \frac{p_n}{q_n} t + \eta_n, t\right) \\
&\eta_\lambda^{\cdot} = \sum_{j=1}^{l} p_{\lambda j}(\mathbf{A}) \xi_j + \mu^{-\sigma} \mathrm{H}_\lambda(\mathbf{A}, \mu^\sigma \xi) + \mu^{1-\sigma}[\alpha_\lambda + V_{1\lambda}(\ldots)] \quad (\lambda = 1, \ldots, k) \qquad (6.11) \\
&\eta_\nu^{\cdot} = \mathrm{H}_\nu(\mathbf{A}, \mu^\sigma, \xi) + \mu[\alpha_\nu + V_{1\nu}(\ldots)] \quad (\nu = k+1, \ldots, n)
\end{aligned}$$

(the arguments in square brackets in the last two groups of equations are clear from the form of (6.10) and (6.11)). Now, using the necessary and sufficient conditions [9] for a $2\pi q$-periodic symmetric solution to exist and employing the same arguments for (6.11) as were used in paragraph 2 of Section 6, we arrive at the conclusion that the following theorem holds.

Theorem 9. In the resonance case (6.9) for sufficiently small $\mu \neq 0$ in system (6.7) an $(l-n+k)$ family of $l-n+k$ quantities from $A_1, \ldots, A_n$ exists, symmetrical about the fixed set $\mathbf{M}$ of $2\pi q$-periodic solutions on the torus

$$\mathbf{u} = \mathbf{A} + \mu \int_0^t \mathbf{U}_1(0, \mathbf{A}, (\mathbf{p}/\mathbf{q})t, t)\,dt + o(\mu), \quad \mathbf{v} = \mathbf{V}(\mathbf{A})t + \mu \int_0^t \mathbf{V}_1(0, \mathbf{A}, (\mathbf{p}/\mathbf{q})t, t) + o(\mu)$$

if

$$\begin{aligned}
&I_\nu(\mathbf{A}) \equiv \int_0^{\pi q} [\alpha_\nu + V_{1\nu}(0, \mathbf{A}, (\mathbf{p}/\mathbf{q})t, t)]\,dt = 0 \quad (\nu = k+1, \ldots, n) \\
&\operatorname{rank} \mathbf{P} = k, \quad \operatorname{rank} \| \partial \mathbf{I}(\mathbf{A}) / \partial \mathbf{A} \|_* = n - k
\end{aligned} \qquad (6.12)$$

(the asterisk denotes a calculation for values of $\mathbf{A}$ which satisfy Eqs (6.12)).

Corollary. If rank $\mathbf{P} = n$, an l-family of $2\pi q$-periodic symmetrical solutions exists in the resonance case (6.9)

4. *The existence of an l-family of periodic solutions*. Consider the 2π-periodic inverse system

$$\mathbf{u}^{\cdot} = \mu \mathbf{U}_1(\mu, \mathbf{u}, \mathbf{v}, t), \quad \mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{u}, \mathbf{v}, t) + \mu \mathbf{V}_1(\mu, \mathbf{u}, \mathbf{v}, t); \quad \mathbf{u} \in \mathbb{R}^l, \quad \mathbf{v} \in \mathbb{R}^n \qquad (6.13)$$

with fixed set $\mathbf{M}$. When $\mu = 0$ we have

$$\mathbf{u} = \mathbf{A}(\text{const}), \quad \mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{A}, \mathbf{v}, t) \qquad (6.14)$$

We will assume that the equation for $\mathbf{v}$ in (6.14) admits of an odd 2π-periodic solution $\mathbf{v} = \varphi(\mathbf{A}, t)$, $\psi(\mathbf{A}, -t) = -\psi(\mathbf{A}, t)$ and we make the following replacement: $\mathbf{u} = \mathbf{A} + \mathbf{p}$, $\mathbf{v} = \psi(\mathbf{A}, t) + \mathbf{q}$. We then obtain

$$\begin{aligned}
&\mathbf{p}^{\cdot} = \mu \mathbf{U}_1(\mu, \mathbf{A} + \mathbf{p}, \psi(\mathbf{A}, t) + \mathbf{q}, t) \\
&\mathbf{q}^{\cdot} = \mathbf{B}_-(\mathbf{A}, t)\mathbf{q} + \mathbf{B}_+(\mathbf{A}, t)\mathbf{p} + \mathbf{Q}_1(\mathbf{A}, \mathbf{p}, \mathbf{q}, t) + \mu \mathbf{V}_1(\mu, \mathbf{A} + \mathbf{p}, \psi(\mathbf{A}, t) + \mathbf{q}, t)
\end{aligned} \qquad (6.15)$$

where the plus (minus) subscript denotes a matrix with even (odd) functions, while $\mathbf{Q}_1$ is a function that is non-linear in $\mathbf{p}$ and $\mathbf{q}$.

We first consider the linear system

$$\mathbf{q}^{\cdot} = \mathbf{B}_{-}(\mathbf{A},t)\mathbf{q} + \mathbf{B}_{+}(\mathbf{A},t)\boldsymbol{\alpha} \quad (\boldsymbol{\alpha} = \text{const}) \tag{6.16}$$

When $\boldsymbol{\alpha} = 0$, system (6.16) is inverse and has only even 2π-periodic solutions (Corollary 3 of Theorem 7), which form a fundamental system $\mathbf{q}^{+}(t,\tau)$; $\mathbf{q}^{+}(t,\tau) = \mathbf{I}_n$ is the identity matrix of the initial conditions (when $t = \tau$). Hence, when $\boldsymbol{\alpha} \neq \mathbf{0}$ the odd solution of system (6.16) has the form

$$\mathbf{q}^{-}(t) = \int_0^t \mathbf{q}^{+}(t,\tau)\mathbf{B}_{+}(\mathbf{A},\ t)\ \boldsymbol{\alpha}\ d\tau$$

According to the results obtained previously [9] the condition rank $\mathbf{q}^{-}(\pi) = n$ is sufficient for the whole family of $\mathbf{A}$ periodic solutions $\mathbf{u} = \mathbf{A}$, $\mathbf{v} = \boldsymbol{\psi}(\mathbf{A}, t)$ of the generating system to be continued over the parameter μ. In this case we have

$$\begin{gathered}\mathbf{u} = \mathbf{A} + \mu[\mathbf{p}^{\circ} + \boldsymbol{\theta}(t)] + o(\mu), \quad \boldsymbol{\theta}(t) = \int_0^t \mathbf{U}_1(0,\mathbf{A},\boldsymbol{\psi}(\mathbf{A},t),t)dt \\ \mathbf{v} = \boldsymbol{\psi}(\mathbf{A},t) + \mu\int_0^t \mathbf{q}^{+}(t,\tau)\{\mathbf{B}_{+}(\mathbf{A},\tau)[\mathbf{p}^{\circ} + \boldsymbol{\theta}(\tau)] + \mathbf{V}_1(0,\mathbf{A},\boldsymbol{\psi}(\mathbf{A},\tau),\tau)\}d\tau + o(\mu)\end{gathered} \tag{6.17}$$

The constants $\mathbf{p}^{\circ}$ are chosen from the condition for the integral for $\mathbf{v}$ to be equal to zero when $t = \pi$. The latter is always possible when rank $\mathbf{q}^{-}(\pi) = n$.

Theorem 10. For sufficiently small $\mu \neq 0$ system (6.13) admits of an l-family of $\mathbf{A}$ symmetric 2π-periodic motions (6.17) if rank $\mathbf{q}^{-}(\pi) = n$.

Note. When $\mu = 0$, system (6.13) can have a periodic solution of the form (6.14) only for fixed $\mathbf{A} = \mathbf{A}^{*}$. In this case, the condition rank $\mathbf{q}^{-}(\pi) = n$ guarantees the existence of only some motions (when $\mathbf{A} = \mathbf{A}^{*}$) of the form (6.17).

Example. Assume that the second-order 2π-periodic system

$$x^{\cdot} = \mu X(\mu,x,y,t), \quad y^{\cdot} = x + \mu Y(\mu,x,y,t) \tag{6.18}$$

is invertible with a fixed set $\{x,y : y = 0\}$. When $\mu = 0$ we have a unique (zero) periodic solution. The matrix $\mathbf{q}^{-}(t)$ consists of one element $q^{-}(t)$ and is calculated by integrating system (6.18) with $\mu = 0$ with initial conditions $x^{\circ} = 1, y^{\circ} = 0$. We have $q^{-}(t) = t, q^{-}(\pi) = \pi \neq 0$. Consequently, in the neighbourhood of zero, system (6.18) has a unique 2π-periodic solution

$$x = \mu\left[x^{\circ} + \int_0^t X(0,0,0,t)dt\right] + o(\mu), \quad y = \mu\int_0^t\left[x^{\circ} + \int_0^{\tau} X(0,0,0,\nu)d\nu + Y(0,0,0,\tau)\right]d\tau + o(\mu)$$

$$\pi x^{\circ} + \int_0^{\pi}\left[\int_0^{\tau} X(0,0,0,\nu)d\nu + Y(0,0,0,\tau)\right]d\tau = 0$$

7. SYSTEMS CLOSE TO INVERSE SYSTEMS

Consider the 2π-periodic system

$$\begin{gathered}\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u},\mathbf{v},t) + \mu\mathbf{U}_1(\mu,\mathbf{u},\mathbf{v},t) \\ \mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{u},\mathbf{v},t) + \mu\mathbf{V}_1(\mu,\mathbf{u},\mathbf{v},t); \quad \mathbf{u} \in \mathbb{R}^{l}, \quad \mathbf{v} \in \mathbb{R}^{n} \quad (l \geqslant n)\end{gathered} \tag{7.1}$$

We will assume that when $\mu = 0$, system (7.1) is inverse with a fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$ and admits of a symmetrical 2π-periodic solution

$$\mathbf{u} = \boldsymbol{\varphi}(t), \quad \mathbf{v} = \boldsymbol{\psi}(t); \quad \boldsymbol{\varphi}(-t) = \boldsymbol{\varphi}(t), \quad \boldsymbol{\psi}(-t) = -\boldsymbol{\psi}(t) \tag{7.2}$$

while the perturbations $\mu\mathbf{U}_1$, $\mu\mathbf{V}_1$ do not belong to the class of inverse systems. We will investigate the problem of the 2π-periodic solutions of system (7.1) when $\mu \neq 0$ when the generating system (with $\mu = 0$) is rough for perturbations of the class of inverse systems.

1. *Roughness in the class of perturbations of general form*. We make the following replacement in (7.1): $\mathbf{u} = \varphi(t) + \mathbf{p}$, $\mathbf{v} = \psi(t) + \mathbf{q}$. As a result we obtain

$$\begin{aligned} \mathbf{p}^{\cdot} &= \mathbf{A}_-(t)\mathbf{p} + \mathbf{A}_+(t)\mathbf{q} + \mathbf{P}(\mathbf{p},\mathbf{q},t) + \mu\mathbf{U}_1(\mu,\varphi(t)+\mathbf{p},\psi(t)+\mathbf{q},t) \\ \mathbf{q}^{\cdot} &= \mathbf{B}_+(t)\mathbf{p} + \mathbf{B}_-(t)\mathbf{q} + \mathbf{Q}(\mathbf{p},\mathbf{q},t) + \mu\mathbf{V}_1(\mu,\varphi(t)+\mathbf{p},\psi(t)+\mathbf{q},t) \end{aligned} \tag{7.3}$$

where the plus (minus) subscript denotes a matrix with even (odd) functions and P and Q are terms non-linear in $\mathbf{p}$ and $\mathbf{q}$.

The fundamental matrix of solutions of the approximation, linear in $\mathbf{p}$ and $\mathbf{q}$, when $\mu = 0$ has the form [9]

$$\mathbf{S}(t) = \begin{Vmatrix} \mathbf{p}^+(t) & \mathbf{p}^-(t) \\ \mathbf{q}^-(t) & \mathbf{q}^+(t) \end{Vmatrix}, \quad \mathbf{S}(0) = \mathbf{I}_{l+n}$$

(I_j is the identity j-matrix). Moreover, if $\mathbf{q}^+(2\pi)$ has no eigenvalues equal to unity, then $\mathbf{S}(2\pi)$ has exactly $l-n$ eigenvalues [14], equal to unity. The following theorem therefore holds.

Theorem 11. If we have $l = n$, $\det(\mathbf{q}^+(2\pi) - \mathbf{I}_n) \neq 0$ in system (7.1), then, for sufficiently small $\mu \neq 0$ in (7.1) a unique 2π-periodic solution exists, which is invertible into the symmetric solution (7.2) when $\mu = 0$.

2. *The case $l > n$*. We will assume, as before, that the matrix $\mathbf{q}^+(2\pi)$ has no eigenvalues equal to unity. Then, in the matrix $\mathbf{S}(t)$ in the last n columns, there are no 2π-periodic solutions [14], while the k-family ($k = l - n$) of 2π-periodic solutions

$$\begin{aligned} p_j &\equiv \varphi_j^*(\mathbf{B},t) = B_1\varphi_{j1}^*(t) + \ldots + B_k\varphi_{jk}^*(t) \quad (j = 1,\ldots,l) \\ q_s &\equiv \psi_s^*(\mathbf{B},t) = B_1\psi_{s1}^*(t) + \ldots + B_k\psi_{sk}^*(t) \quad (s = 1,\ldots,n) \end{aligned} \tag{7.4}$$

is symmetric with respect to the set $\{\mathbf{p},\mathbf{q} : \mathbf{q} = 0\}$. If we introduce a small parameter $\varepsilon = \mu^\sigma$ into system (7.3) by means of the replacement $(\mathbf{p}, \mathbf{q}) \to (\varepsilon\mathbf{p}, \varepsilon\mathbf{q})$ and use the system $\{\theta^-_{\alpha\nu}(t), \chi^+_{\beta\nu}(t)\}$ of 2π-periodic solutions of the conjugate linear system, the equations for $B_1, \ldots, B_k$ have the form

$$\begin{aligned} B_\nu^{\cdot} &= \varepsilon^\lambda F_\nu(\varepsilon,\mathbf{p},\mathbf{q},t) + \varepsilon^{-1}\mu F_{1\nu}(\mu,\varphi(t)+\varepsilon\mathbf{p},\psi(t)+\varepsilon\mathbf{q},t) \quad (\nu = 1,\ldots,k) \\ F_\nu &\equiv \varepsilon^{-1}\left[\sum_{\alpha=1}^{l} P_\alpha(\varepsilon\mathbf{p},\varepsilon\mathbf{q},t)\theta^-_{\alpha\nu}(t) + \sum_{\beta=1}^{n} Q_\beta(\varepsilon\mathbf{p},\varepsilon\mathbf{q},t)\chi^+_{\beta\nu}(t)\right] \\ F_{1\nu} &\equiv \sum_{\alpha=1}^{l} U_{1\alpha}(\mu,\varphi(t)+\varepsilon\mathbf{p},\psi(t)+\varepsilon\mathbf{q},t)\theta^-_{\alpha\nu}(t) + \\ &+ \sum_{\beta=1}^{n} V_{1\beta}(\mu,\varphi(t)+\varepsilon\mathbf{p},\psi(t)+\varepsilon\mathbf{q},t)\chi^+_{\beta\nu}(t) \quad (\lambda \in \mathbb{N},\ \lambda \geqslant 1) \end{aligned} \tag{7.5}$$

Further analysis requires the setting up of the amplitude equation and the determination of its simple roots. The two cases of Section 5 are possible here. We will consider only one of these cases below.

Suppose solution (7.2) belongs to the k-family or 2π-periodic solutions $\mathbf{u} = \varphi(\mathbf{A}, t)$, $\mathbf{v} = \psi(\mathbf{A}, t)$. In this case the amplitude equation has the form

$$\int_0^{2\pi} \mathbf{F}_1(0,\varphi(\mathbf{A},t),\psi(\mathbf{A},t),t)dt = \mathbf{0} \tag{7.6}$$

Theorem 12. For sufficiently small $\mu \neq 0$, to each simple root of amplitude equation (7.6) there will correspond a unique 2π-periodic solution of system (7.1), which differs from the symmetrical solution by a quantity $O(\mu)$.

Note. Amplitude equation (7.6) corresponds to case (5.7). It is obviously easy to write amplitude equations in the cases corresponding to (5.8)–(5.10) also.

3. *The case rank* $q^{-}(\pi) = n$. In this case in the part of system (7.3) that is linear in p and q when $\mu = 0$ there is an $(l - n)$-family of 2π-periodic solutions, symmetric with respect to the set $\{\mathbf{p}, \mathbf{q} : \mathbf{q} = \mathbf{0}\}$. When rank $\mathbf{p}^{-}(\pi) = n$ we arrive at case 2. If rank $\mathbf{p}^{-}(\pi)\ \varkappa < n$, this linear system also has an $(n - \varkappa)$-family of 2π-periodic solutions symmetric with respect to the set $\{\mathbf{p}, \mathbf{q} : \mathbf{q} = \mathbf{0}\}$. Using the periodic solutions of the conjugate linear system, we arrive at a system of the form (7.5) in which ν changes from 1 to $1 - \varkappa$.

Hence, only when $1 = n$ can the inverse generating system be rough in the sense of the property of having periodic motion if perturbations of general form are considered. Note also that the most degenerate cases, when the inverse generating system is not rough even for perturbations from the class of inverse systems, is investigated similarly for perturbations of general form, like the cases considered here. Here it is necessary to use well-known results [14] on the investigation of inverse systems and also those described in Section 5. These cases are omitted due to shortage of space.

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